Sample Creation

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In this appendix we discuss how we can create samples for the optimization. The samples include liability amounts $\xi \times T_0$ and $\varepsilon \times T_0$, expected return rates α of equities, factor rates φ and expected return rates R of bonds.

We have divided this appendix into 5 sections:

1) How to create samples of the liability amounts.
2) How to create examples of the expected return rates of equities and factor rates.
3) How to create samples of the expected return rates of bonds if we treat them as equities.
4) The twist model for bonds if there is no correlation between bonds and equities or factors.
5) How we can include correlation between bonds, equities, and factors.

1. <u>Liability Amounts</u>

Note that since each ξ_{ti} and ξ_t is independent of other random variables, we can create samples of them independently. We generate the amounts $\xi_{ti} \times T_0$ and $\varepsilon_t \times T_0$ instead as the distributions of them are fixed but not those of ξ_{ti}, ξ_t.

In most case ξ_{ti} and ξ_t vary according to different T_0. Assuming that $\xi_{ti} \times T_0$ and $\varepsilon_t \times T_0$ follow normal distributions $N(\mu, \sigma^2)$, they can be generated easily with a $N(0,1)$ generator and a transformation

$x = \mu + z \times \sigma$

$\uparrow \qquad \uparrow$ *random* $N(0,1)$

random $N(\mu, \sigma^2)$

2. <u>Expected return rates of equities and factor rates</u>

We assume that $(\alpha_{t1}, \alpha_{t2}, \ldots, \alpha_{tn}, \varphi_{t1}, \ldots \varphi_{tm})$ follow a joint multivariate normal distribution. $N_{m+n}(\mu, C_t)$ where $\mu_t =$ mean vector

and $C_t =$ var-cov matrix.

Random samples of $(\alpha_{t1}, \alpha_{t2}, \dots, \alpha_{tn}, \varphi_{t1}, \dots \varphi_{tm})$ can be generated with a $N_{m+n}(\underset{\sim}{\mu}, C_t)$ generator by inputting $\underset{\sim}{\mu}_t$ and C_t.

3. <u>Expected return rates of bonds by treating them as equities.</u>

   a) Roll the bond after maturity.
   We treat the bonds as equities by assuming that the expected return rates of different periods are independent. ie. we generate
   $\{\alpha_{11}, \alpha_{12}, \dots, \alpha_{1n}, R^0_{1,1}, R^0_{1,2}, \dots, R^0_{1,30}, \varphi_{11}, \dots \varphi_{1m}\}$
   $\{\alpha_{t1}, \alpha_{t2}, \dots, \alpha_{tn}, R^{t-1}_{t,1}, R^{t-1}_{t,2}, \dots, R^{t-1}_{t,30}, \varphi_{t1}, \dots \varphi_{tm}\}$
   using the multivariate normal distribution.

   We also assume that after a bond matures, we simply roll it. Hence,
   $R^{t-1}_{t,1} = R^{t-1}_{t,2} = \cdots = R^{t-1}_{t,t-1}$.

   b) <u>Using the Buy-Hold model.</u>

   We generate $E(R^0_{1,m}), \dots, E(R^{m-1}_{m,m})$ as in (a). However, instead of generating $E(R^{t-1}_{t,m})$ for $t > m$ and consider the bond as an equity, we calculate $E(R^0_{t,m})$ for $t \le m$ and set $E(R^0_{t,m}) \equiv 0$ for $t > m$. These are all the samples we need in the buy-hold model.

   Note: The problem with these two approaches is that
   $1 + E(R^0_{m,m}) = (1 + E(R^0_{1,m}))(E(R^1_{2,m})), \dots, (1 + E(R^{m-1}_{m,m}))$ is not a constant since the variables on the right hand side are independent. One way to get around this is to replace the values of $E(R^0_{m,m})$ by a constant $(1 + y_{0m})^m - 1$ where y_{0m} is the initial yield rate of bond maturing at time *m*.

4. <u>**Twist Model**</u>

We generate the yield curves in the future so that we can calculate the corresponding sample values of $R^0_{t,m}$ for $t \le m$.
Let $y_{s,t} =$ yield rate of a bond maturing at time *t*, s periods from now. First we generate the change of short rates using some model such as CIR:

$$\Delta r_t = a(b - r_t)\Delta t + \sigma\sqrt{r_t}\sqrt{\Delta t}\ \hat{n}$$

where $\hat{n} \sim N(0,1)$ can be generated as usual.

Consider the shift and twist models for yield curves:

Shift: $\frac{dy_{t,s}}{dx} = -(1 + y_{t,s})$ for all s.

Twist: $\frac{dy_{t,s}}{dz} = -(1 + y_{t,s})(s - mid)$ for all s.

In the discrete approximation we have:

Shift: $y_{t+\Delta,s} = y_{t,s} + [-(1 + y_{t,s})]\Delta x$
x is a process related to r.

Twist: $y_{t+\Delta,s} = y_{t,s} + (1 + y_{t,s})(s - mid)\Delta z$
z is a process related to r.

We assume that Δx_t and Δz_t follow some joint distribution and combining, we have:
$y_{t+\Delta,s} = y_{t,s} + [-(1+y_{t,s})]\Delta x_t + (1+y_{t,s})(s-mid)\Delta z_t - (*)$ for all s.

From the above equation, given today's yield curve:

$y_{0,0}, y_{0,\Delta}, y_{0,2\Delta}, \ldots, y_{0,30}$ we can calculate $y_{\Delta,\Delta}, y_{\Delta,2\Delta}, y_{\Delta,3\Delta}, \ldots, y_{\Delta,30}$ the yield curve at time Δ. Iteratively, we get the yield curves at time $\Delta, 2\Delta, \ldots, 30$.

Now, we have 3 different distributions $\Delta x_t, \Delta z_t$ and Δr_t. but only two degrees of freedom. However, we also have the following constraint:

$$y_{t+\Delta,t+\Delta} = y_{t,t} + \Delta r_t \quad (1)$$

sine $y_{t,t} \approx r_t$

From $(*)$, considering $s = t + \Delta$:

$$y_{t+\Delta,t+\Delta} = y_{t,t+\Delta} + [-(1+y_{t,t+\Delta})]\Delta x_t + (1+y_{t,t+\Delta})(t+\Delta-mid)\Delta z_t$$

Note that at time t, all the y's in the equation are known, so that we have a relation of $\Delta r_t, \Delta x_t, \Delta z_t$ and

$$\Delta x_t = -\frac{y_{t,t} - y_{t,t+\Delta} + \Delta r_t}{1 + y_{t,t+\Delta}} + (t+\Delta-mid)\Delta z_t \quad (**)$$

Now samples of Δr_t can be generated by the CIR model, we have to generate samples of Δz_t, however, it is not completely developed as of 10/31/1999. We shall look into this in the future.

Theoretically, it is very nice up to this point. However, we are only interested in $y_{i,j}$ in terms of r's and z's for integers *i,j*. We can use brute force to calculate them but it is very time consuming for small Δ.

We can also take $\Delta = 1$. Let

$$\Delta r_t = r_t - r_{t-1}$$
$$\Delta z_t = z_t - z_{t-1}$$
$$\Delta x_t = x_t - x_{t-1}$$

$\therefore$ (*) and (**) $\rightarrow$

$$E(y_{t,s}) = E(y_{t-1,s}) + (1 + E(y_{t-1,s}))\left[(s-t)E(\Delta z_t) + \frac{E(\Delta r_t) - E(y_{t-1,t}) + E(y_{t-1,t-1})}{1 + E(y_{t-1,t})}\right]$$

$$= E(y_{t-1,s}) + (\varphi_{s,t}E(\Delta z_t) + \psi_{s,t}E(\Delta r_t) + \frac{(1 + E(y_{t-1,s})E(y_{t-1,t-1}) - E(y_{t-1,t})}{1 + E(y_{t-1,t}}$$

$$where \quad \varphi_{s,t} = (1 + E(y_{t-1,s}))(s-t)$$

$$\psi_{s,t} = \frac{1 + E(t_{t-1,s})}{1 + E(y_{t-1,t})}$$

So, in this simplified model, we need only know $y_{0,1}, y_{0,2}, \ldots, y_{0,30}$ to generate $y_{t,t}, \ldots, y_{t,30}$ Once the yield curves are known, samples of the expected return rate can be generated as follows:

$$1 + R^0_{t,m} = \frac{(1 + y_{0m})^m}{(1 + y_{tm})^{m-t}}$$

5. <u>Correlation between bonds, equities, and factors</u>

If we consider bonds as equities, then we can have correlations as usual. However, if we are using a bond model, we need to refine the movement of the expected return rates of equities, and factors so that they are comparable to the movement of short rates. In our original model, the return rates are updated each period (year). However, the short rates are updated every Δt. Due to the mean reversion property, we should not change Δt to 1. So, we should update the return rates every Δt, so that it can be correlated with the short rates after one period.

ie. $(1+\alpha_{1,1}) = (1+\alpha_{\Delta t,1}),(1+\alpha_{2\Delta t,1}),\ldots,(1+\alpha_{1-\Delta t,1})$.

We have developed some extension of this in other articles.