Objective Function

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The Objective Function that is briefly described in the article "Asset Allocation with Liabilities – Redefining Risk" is expanded to account for risks as well as goals. It briefly demonstrates how this approach can account for the mean variance approach, the Allais paradox and regret.

In our approach, we are using goals or objectives so we will seek to maximize the likelihood of achieving those goals (or minimize the likelihood of risk). Therefore we will use the Lagrangian max-min approach.

Section I: Building the Objective Function

We have redefined risk as the possibility and consequences of failing to achieve one's objectives. In order to accomplish this mathematically, we set balances to the portfolio called B_n, which act as targets or reference values. Each has a probability P assigned to it for a time period t.

In order to derive this, recall that the Lagrangian with an inequality constraint is defined as:

$$L(x,\lambda) = f(x) + \sum_{i=1}^{n} \lambda_i (b_i - g_i(x)) \qquad \textbf{[obj 1.0]}$$

where the function *f*(x) is the expected return whose value we want to maximize, subject to the constraints $((g_1(x), g_2(x), \ldots, g_n(x)) \leq (b_1, b_2, \ldots b_n)$, and λ_i is a weighting factor.

Consider a single period where T_0 is the value of the portfolio at the beginning of the period and T_1 is the value at the end. Substituting our notation into the Lagrangian we have:

$$L(T_1,\lambda) = E(T_1) + \sum_{i=1}^{n} \lambda_i (P_i - F(B_i)) \qquad \textbf{[obj 1.1]}$$

where F( ) is the cumulative density function: $F(B_i) = \Pr(T_1 \leq B_i)$. The weighting factor will be called the "Objective Weight Factor" (OWF).

Objectives

Our definition focuses on objectives as opposed to down side limits. This means that risk is now relative to particular goals as opposed to falling below a threshold value. It allows us to accommodate the simultaneous purchase of insurance and a lotto ticket in the utility function. The downside protection of insurance and the upside potential of the lotto ticket are two separate goals.

So far, our formulation has focused only on finding the lowest probability of falling below a certain value. We now extend this to achieving other goals, namely increasing the probability of exceeding a certain value. Such a Lagrangian would be of the form:

$$L(x,\lambda) = f(x) + \sum_{i=1}^{n} \lambda_i (b_i + g_i(x)) \qquad \textbf{[obj 1.2]}$$

where the function *f*(x) is the expected return whose value we want to maximize, subject to the constraints $((g_1(x), g_2(x), \ldots, g_n(x)) \geq (b_1, b_2, \ldots b_n)$.

The cumulative density function is then changed to $F'(B_i) = \Pr(T_1 \geq B_i)$.

Therefore, the new Lagrangian that includes both downside risk and upside goals becomes:

$$L(T_1,\lambda) = E(T_1) + \sum_{i=1}^{n} \lambda_i (P_i - F(B_i)) + \sum_{i=n+1}^{m} \lambda_i (P_i + F'(B_i)) \qquad \textbf{[obj 1.3]}$$

Multiperiod

So far, our utility function covers only one time period. It is fairly straightforward to extend this into multiple time periods by adjusting the Lagrangian as follows:

$$L(T_F,\lambda) = E(T_F) + \sum_{t=1}^{F} \left(\sum_{i=1}^{n} \lambda_{it} (P_{it} - F_t(B_{it})) + \sum_{i=n+1}^{m} \lambda_{it} (P_{it} + F_t'(B_{it})) \right) \qquad \textbf{[obj 1.4]}$$

where $F_t(B_{it}) = \Pr(T_t \leq B_{it})$.

Mean-Variance

For mean-variance we can use the same technique, but with some adjustments. Since we are dealing in expected returns as opposed to total portfolio values, we must normalize:

$$T_1 \to \left(\frac{T_1}{T_0} - 1\right)$$

Recall Telser's criterion, which involves maximizing the expected return of a portfolio subject to the probability that the actual return is less than some minimum acceptable return being lower than some predetermined level. It is expressed as:

Maximize $\quad E(\overline{R}_p)$

Subject to the constraint: $\quad \text{Prob}(R_P \leq R_L) \leq \alpha$

This is very similar to what we've been discussing.

If we assume normality, then Telser's criterion becomes a straight line in mean-variance space. It is:

$$\overline{R}_P = R_L + \sigma_p F(\alpha)$$

where F(α) has been limited to the critical value of the standard normal distribution associated with the probability level alpha. We have shown elsewhere that the optimal portfolio will occur at the intercept between this line and the M-V efficient set.

Therefore, Telser's criterion in our Lagrangian notation becomes:

$$L(\overline{R}_p, \lambda) = E(\overline{R}_p) + \lambda(P - F(\alpha)) \qquad \textbf{[obj 1.5]}$$

which is the probability of the portfolio falling below the floor at each time period t. Where this definition differs from Telser's approach is that we are not limited by the number of floors we set for any time period. Therefore, we could use a ladder of floors, with each a different probability. The floors, with their associated probabilities, act as tolerances for short and long term volatility, catastrophic events, and can also be used as a means of setting the sponsor's confidence in their forecasting ability. Risk is now defined by multiple variables instead of just one.

Allais Paradox

There are two other extensions that we will briefly discuss. One is to accommodate the Allais Paradox, and the other the concept of regret.

The Allais Paradox demonstrates a contradiction in investor behavior. We can eliminate this contradiction, if we assume non-linearity in the investor's change in their Objective Weight Factor. Therefore, our Multi-Period Lagrangian becomes:

$$L(T_F, \lambda, \theta) = E(T_F) + \sum_{t=1}^{F} \left(\sum_{i=1}^{n} \theta_{it}(\lambda_{it})(P_{it} - F_t(B_{it})) + \sum_{i=n+1}^{m} \theta_{it}(\lambda_{it})(P_{it} + F_t'(B_{it}))\right) \qquad \textbf{[obj 1.6]}$$

where $F_t(B_{it}) = \Pr(T_t \leq B_{it})$, λ_{it} is the Objective Weight Factor, and θ_{it} is the Objective Weight Factor Function (OWF Function).

Regret

Regret is more difficult to include in this methodology. The traditional solution is to restructure the utility function to form a "regret/rejoice" function for pairwise lotteries that contains the outcomes of both the chosen and the foregone lotteries. One then seeks to maximize a function of the form: r(x, y) = u(x) - u(y), where u(x) is the chosen strategy and u(y) is the rejected strategy. We then have a maximize "rejoice" and minimize "regret" utility.

In order to incorporate this concept into our methodology, we will divide “regret” into two elements or types. The first is the regret experienced when a portfolio goes up and then down. We call this Type I regret. This is the regret one feels for having not sold an asset when it was doing well. The second is the regret experienced in not having invested in a superior performing portfolio. This we will call Type II regret. It is the regret one feels for not having purchased the portfolio in the first place.

We can approximate Type I regret behavior by using two methods. One method is by setting a volatility constraint at each time period and the other is by setting a rebalancing strategy. The setting of a volatility constraint is essentially traditional mean-variance analysis where the “Risk Aversion Multiplier” times the standard deviation of returns is used to define risk. Therefore, by setting a maximum (downside) volatility at each time period we infer our investor’s tolerance for Type I regret.

Notice that at one point a risk aversion multiplier and a rebalancing strategy are the same thing.

Type II regret is extremely difficult to address with our method. Since there are thousands of investment possibilities confronting our investor, there will always be superior performing assets that were not selected. To some degree, over diversification of portfolios may be a symptom of type II regret (and not just misunderstanding systematic risk).

Type II regret involves minimizing regret. Regret is then the probability that another portfolio’s return Y_t is greater than our portfolio’s return T_t. The greater the difference, the greater the regret. Mathematically, this can be described as:

$$\text{Minimize } \gamma_t(x_t)[\Pr(Y_t \geq [T_t + x_t])]$$

where γ_t is a positive function of x_t.

It is simple to show that this can be mapped into the constraint:

$$\text{Maximize } \gamma_t(B_t)\Pr(T_t \geq B_t)$$

which is;

$$\text{Maximize } \gamma_t(B_t)F'(B_t) \qquad \textbf{[obj 1.7]}$$

Part II – Solving the Probability Function

In order to solve our Lagrangian, we must find the solution for the constraints of the form:

$$P(T_t \le B_{1t}) \le P_{1t}$$
$$\vdots$$
$$P(T_t \le B_{nt}) \le P_{nt}$$

These probabilities can be found by solving the following:

$$P(T_t \le B_{it}) = \int_{-\infty}^{B_{it}} f(z)dz \qquad \text{[obj 1.8] } \mathbf{[2.10]}$$

where $f(z)$ is the probability density of T_t.

The problem with the integral in [2.10] is that we don't know the density function *f(z)* explicitly. In fact, it is an extremely complex function. In order to find its solution, we must <u>build</u> and solve the function numerically.

Building the Density Function

Stochastic Variables

In order to solve this probability function, we must first determine the probability density function for our stochastic process. The first thing then is to determine the stochastic variables involved in the problem.

<u>Assets</u>

Each asset has an expected return attached to it. Note that for bonds can involve an additional mapping (if one were to use a one or two factor bond model). The bond model factors would then be the stochastic variable and our asset return would be a function of those factors (this is why we've left the start time k in our definition). The expected returns on our assets can be divided into the returns and weights for individual assets.
α_{ts}^{k} is the return of asset *s* at time *t* with start time *k*.

<u>Liabilities</u>

Liabilities are described as a linear summation of factors and the liability's exposure to these factors.
φ_{ts} is the liability factor return for factor *s* at time *t*.
ξ_{ts} is the liability factor weight for factor *s* at time *t*

ε_t is the residual liability weight at time *t*.

Liability factors are selected based on a number of conditions. One of these conditions is that they be a priori factors. That is, even though the factor returns are uncertain, the factor exposures must be known a priori – at the beginning of the period. Another condition is that of relevance. The factor must have a non-trivial correlation with at least one of the asset returns. Also, we must be able to forecast the factor's rate of return. There is no point in selecting a factor where we have no ability to forecast its behavior. Finally, we attach an independence condition in order to add a degree of rigor to the selection of factors. If independent (or near independent) factors cannot be selected, then the exposure should be left in the residual term.

Examples of φ_{ts} would be:

φ_{t1} inflation rate over period t

φ_{t2} real (after inflation) GDP growth over period *t*.

φ_{t3} sector GDP rate normalized to real GDP.

ε_t residual term.

Most cases will likely involve inflation and the residual. The factor weights ξ_{ts}, are assumed to be independent of the factor returns.

We then have n assets and m liabilities. Therefore T_t is a function of the (n+2m+1) known random variables:

$$(\alpha_{11},\ldots,\alpha_{1n},\varphi_{11},\ldots,\varphi_{1m};\cdots;\alpha_{t1},\ldots,\varphi_{tm};\xi_{11},\ldots,\xi_{tm};\varepsilon_1,\ldots,\varepsilon_t)$$

Density Function

So, we can calculate $P(T_t \le B_{it})$ through the distributions of these variables.

More generally, suppose $Y = g(Z)$ and $f_Z(\mathbf{z})$ is the density of Z.

Then $P(Y \le B) = P(g(Z) \le B)$

$$= \int 1_{\{g(Z)\le B\}}(\mathbf{z})\, f_Z(\mathbf{z})\, d\mathbf{z} \qquad \textbf{[obj 1.9]}$$

where

$$1_{\{g(Z)\le B\}}(\mathbf{z}) = 1 \text{ if } g(\mathbf{z}) \le B$$
$$=0 \text{ if } g(\mathbf{z}) > B$$

Note that signs are reversed for goals.

Setting $Z=(\alpha_{11},\ldots,\alpha_{1n},\varphi_{11},\ldots,\varphi_{1m};\cdots;\alpha_{t1},\ldots,\varphi_{tm};\xi_{11},\ldots,\xi_{tm};\varepsilon_1,\ldots,\varepsilon_t)$, $Y=g(Z)=T_t$ and $f_{n,m,t}(\mathbf{z})$ as the joint density function of Z, we have,

$$P(T_t \le B_{it}) = \int_{R^{(n+2m+1)t}} \int 1_{\{T_t \le B_{it}\}}(\mathbf{z}) f_{n,m,t}(\mathbf{z}) d\mathbf{z}$$ **[obj 1.10] [2.11]**

We now have sufficient information to find a solution. To do this, we must find a generalized density function $f_{n,m,t}(\mathbf{z})$ for equation [2.11]. This function is the product of all the density and joint density functions of the random variables α,ξ_{tm},φ and ε_t. Therefore, it includes a rather large variance-covariance matrix, which captures the relationships between the asset classes and liability factors for each point in time. First we determine the generalized density function, then we solve for the indicator function in terms of E(T).

Determining a Generalized Density Function

To determine $f_{n,m,t}(\mathbf{z})$ we will make the following assumption:

The following classes are independent with each other.

$$\begin{Bmatrix} \alpha_{11}, \alpha_{12}, \ldots, \alpha_{1n} \\ \varphi_{11}, \varphi_{12}, \ldots, \varphi_{1m} \end{Bmatrix} \perp \ldots \perp \begin{Bmatrix} \alpha_{t1}, \alpha_{t2}, \ldots, \alpha_{tn} \\ \varphi_{t1}, \varphi_{t2}, \ldots, \varphi_{tm} \end{Bmatrix} \perp \{\xi_{11}\} \perp \ldots \perp \{\xi_{tm}\} \perp \{\varepsilon_1\} \perp \ldots \perp \{\varepsilon_t\}$$

and $\varphi_{i1} \perp \varphi_{i2} \perp \ldots \perp \varphi_{im}$ **[obj 1.11] [2.1]**

Each variable follows a Markov process, therefore it is not path dependent. Therefore they are independent from one time period to the next and thus have intertemporal orthogonality. By definition, the residual variable epsilon and the liability factor returns are independent of each other as well as the liability factor weights. Thus, the independencies arise from the Markov nature of the variables as well as the method in which the factors are determined.

Notice that we have not divided the asset returns into a stochastic variable and a residual, but have instead left them as single stochastic variables. This is fine if we can easily determine a density function (and in many cases a joint density function) for the variable. However, in cases where the asset returns are non-normal and we cannot find a suitable non-normal joint density function, we can divide the asset return into two random variables of the form:

$$\alpha = \beta \cdot G + \delta$$ **[obj 1.12] [2.1a]**

Where, the beta component behaves normally (or with a known joint distribution) and has covariances with other assets, while the delta variable has the residual random behavior (and G is a constant or some function).

Since there are correlations between the asset return rates and factor rates, we have to assume they follow some type of joint distribution. Suppose the joint density of $(\alpha_{i1},\ldots,\alpha_{in},\varphi_{i1},\ldots,\varphi_{im})$ is $q_i(\mathbf{y}_i)$ and the densities of ξ_{ij} is $p_{ij}(x_{ij})$, and ε_i is $p_i(x_i)$. From the first assumption we can see that the distribution of ξ_{ij} and ε_i can be modeled independently, so we can choose $p_{ij}(x_{ij})$ and $p_i(x_i)$ easily. But the joint density is not as trivial, since we have to capture the correlations between the random variables.

Since we consider $(\alpha_{i1},\ldots,\alpha_{in},\varphi_{i1},\ldots,\varphi_{im})$ as one distribution, the mean is a vector.

$$C_i = \text{variance – covariance matrix of } \begin{pmatrix} \alpha_{i1} \\ \vdots \\ \alpha_{in} \\ \varphi_{i1} \\ \vdots \\ \varphi_{im} \end{pmatrix} \qquad \text{[obj 1.13]} \ \textbf{[2.2]}$$

$$= \left(\begin{array}{ccc|ccc} \mathrm{var}(\alpha_{i1}) & \mathrm{cov}(\alpha_{i1},\alpha_{i2}) & \cdots & \mathrm{cov}(\alpha_{i1},\varphi_{i1}) & \cdots & \mathrm{cov}(\alpha_{i1},\varphi_{im}) \\ \vdots & \vdots & \vdots & & & \\ \cdots & \cdots & \mathrm{var}(\alpha_{in}) & \mathrm{cov}(\alpha_{in},\varphi_{i1}) & \cdots & \mathrm{cov}(\alpha_{in},\varphi_{im}) \\ \hline & & & \mathrm{var}(\varphi_{i1}) & 0 & 0 \\ & & & 0 & \mathrm{var}(\varphi_{i2}) & 0 \\ & & & 0 & 0 & \mathrm{var}(\varphi_{im}) \end{array}\right) \quad \text{[obj}$$

$$= 0 \quad as \quad \varphi_{i1}\perp\cdots\perp\varphi_{im}$$

1.14]**[2.3]**

The non-diagonal elements in the bottom right quadrant are zero, since $\varphi_{i1}\perp\ldots\perp\varphi_{im}$.

The joint density of $(\alpha_{11},\ldots,\alpha_{1n},\varphi_{11},\ldots,\varphi_{1m};\cdots;\alpha_{t1},\ldots,\varphi_{tm};\xi_{11},\ldots,\xi_{tm};\varepsilon_1,\ldots,\varepsilon_t)$ is just the product of the independent densities. Once we have determined $p_{ij}(x_{ij})$, $p_i(x_i)$, and $q_i(\mathbf{y}_i)$. By the orthogonality assumption we obtain:

$$f_{n,m,t}(\mathbf{z}) = q_1(\mathbf{y}_1)\cdots q_t(\mathbf{y}_t)\cdot p_{11}(x_{11})\cdots p_{tm}(x_{tm})\cdot p_1(x_1)\cdots p_t(x_t) \ \text{[obj 1.15]} \ \textbf{[2.4]}$$

$$\uparrow \ (n+2m+1)t \ \text{variables.}$$

We must still select a joint density, however.

<u>Normal Distribution</u>

Since a well known one is the multivariate normal distribution we will use this for now. Therefore:

$$(\alpha_{i1},\ldots,\alpha_{in},\varphi_{i1},\ldots,\varphi_{im})^T \sim N_{n+m}(\boldsymbol{\mu},C_i) \qquad \text{[obj 1.16] } \textbf{[2.5]}$$

where $\mu_i = (E(\alpha_{i1}),\ldots,E(\varphi_{im}))^T$ is the mean vector

The joint density of $\mathbf{y}_i = (\alpha_{i1},\ldots,\alpha_{in},\varphi_{i1},\ldots,\varphi_{im})^T$ for a normal distribution becomes:

$$q_i(\mathbf{y}_{i,}) = \frac{1}{(2\pi)^{\frac{n+m}{2}} \left|\det C_i\right|^{\frac{1}{2}}} \times \exp\left\{-\frac{1}{2}(\mathbf{y}-\mu)^T C_i^{-1}(\mathbf{y}_i-\mu)\right\} \qquad \text{[obj 1.17] } \textbf{[2.6]}$$

Notice in [2.6] that we have assumed that we know the explicit function for the mean vector. This function is non-trivial however. See the article "Building the Expectation Function" for a detailed explanation of how this function is constructed.

The multi-variate normal distribution has the advantage of having an easily solvable joint density function. It is not necessary, however, for the return variable to follow a normal distribution. By using a function such as [2.1a] or in the case of bonds a single or multifactor model, we can capture non-normal stochastic processes. Any covariances between those processes would be based on a joint normal distribution that was embedded in them.

Solving the Integral

From [2.10] and [2.4] we must solve the integral

$$P(T_t \le F) = \int_{-\infty}^{F} f(z)dz$$

where $f(z)$ is the density of T_t of the form:

$$f_{n,m,t}(\mathbf{z}) = q_1(\mathbf{y}_1)\cdots q_t(\mathbf{y}_t)\cdot p_{11}(x_{11})\cdots p_{tm}(x_{tm})\cdot p_1(x_1)\cdots p_t(x_t)$$

$\uparrow$ $(n+2m+1)t$ variables.

This integral can be massive. For thirty time periods with one liability factor return and seven assets the integral will have 300 dimensions. The easiest way to determine a solution is through a Monte Carlo method.

<u>Monte Carlo Method</u>

Under a common Monte Carlo method, the integral can be solved by generating a series of points across the range $(-\infty, F)$. Call these points $t_1, t_2, \ldots, t_n$. The probability is calculated using the following approximation.

$$P(T_t \leq F) \approx f_{(t_1)} + \cdots + f_{(tn)}$$

This method holds by the law of large numbers. The problem lies in that we do not know the generalized density function f_t.

We get around this problem by using the following method:

Suppose we can somehow obtain the 'real samples' for T_t. [In traditional Monte Carlo methods, samples are taken at random (evenly distributed) from $(-\infty, F)$. 'Real samples' lie along the correct or 'real' distribution.]

Therefore, by using 'real samples' we can approximate the value for P, by satisfying the following:

$$P(T_t \leq F) \approx \frac{\#\, of\ T_t's\ (real\ samples) \leq F}{n}$$

The problem is that we must determine the 'real samples'. This is not difficult. We simply take a series of samples from our random variables $\varphi, \alpha, \xi, \varepsilon$ and find their T_t equivalent.

Therefore,

$$P(T_t(\alpha, \varphi, \xi, \varepsilon) \leq F) \approx \frac{\#\, of\ T_t(\alpha, \varphi, \xi, \varepsilon) \leq F}{n}$$

Therefore we can find a solution as long as we know how to generate $\varphi, \alpha, \xi, \varepsilon$ samples. We need only know the mean, variance and covariance matrix of the joint distribution of $(\alpha_{t1}, \ldots, \alpha_{tn}, \varphi_{t1}, \ldots, \varphi_{tm})$ as well as the compounding process (which we have constructed in “Building the Expectation Value”) so that we can compute the T_t equivalent.

Appendix

Variable Definitions:

T_t	the total portfolio value at time t.
ω_{ts}	the weight of asset s at time t.
α_{ts}^{k}	the return of asset s at time t with start time k.
L_t	the liability at time t.
φ_{ts}	the liability factor return for factor s at time t.
ξ_{ts}	the liability factor weight for factor s at time t.
ε_t	the residual liability weight at time t.
A_t^k	the return of all the assets at time t where k is the start time.
R_{ij}	return rate of a bond maturing at j as of time i. This rate of return is from i-1 to i.
R_{tm}^{bk}	rate of return of the b^{th} bond asset class, maturing at m as of time t with an initial start date k.
P_{tm}	price of a pure discount bond with maturity at time m as of time t.
N_b	the maximum number of years considered for the bth bond asset class.