

# Building the Expectation Function

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*The Expectation Function that is briefly described in the article “Asset Allocation with Liabilities – Redefining Risk” is expanded to account for non-perpetual assets (maturing bonds) and different rebalancing and reinvestment mechanisms.*

The Expectation Function described in the article “Asset Allocation with Liabilities – Redefining Risk” was of a general form. It was only applicable for perpetual investments such as equities and could not accommodate non-perpetual investments such as maturing bonds. Once a maturity event occurs, the issue of re-investment also becomes relevant. This article builds the generalized expectation function for the following strategies:

- a) Buy and Hold Perpetual (Perpetual Only)
- b) Buy and Hold Maturing (Perpetual and Maturing)
- c) Constant Mix Perpetual (Perpetual Only)
- d) Constant Mix Maturing (Perpetual and Maturing)

## General Expectation Function

We begin by including our liability forecast directly into our expected return. Rather than viewing future liabilities as negative assets, we consider them to be negative returns on our existing assets. They are future obligations that are met with the future returns on existing assets as well as cash from other sources (dues, premiums or other income). Therefore, the total market value of our portfolio at each time period becomes the compounded return of our assets less the reductions due to our liabilities. The expected value of the portfolio then becomes:

$$E(T_1) = [1 + E(A_1^0)]E(T_0) - E(L_1)T_0$$

$$E(T_2) = [1 + E(A_2^1)]E(T_1) - E(L_2)T_0$$

⋮

$$E(T_t) = [1 + E(A_t^{t-1})]E(T_{t-1}) - E(L_t)T_0 \quad [1.1]$$

Where,  $T_t$  is the total portfolio value at time  $t$ ,  $L_t$  is the liability at time  $t$  and  $A_t^k$  is the return of all the assets at time  $t$  where  $k$  is the start time. Notice that for the assets  $E(T_t)$  is dependent upon  $E(T_{t-1})$  while for the liabilities,  $E(T_t)$  is dependent upon  $T_0$ . Since we are in the present,

$T_0$  is does not require an expectation, rather it is a known variable. Therefore, we want to express [1.1] in terms of  $T_0$ .

### Assets

The expected returns on our assets can be divided into the returns and weights for individual assets. We do this in a way that directly captures the effect of compounding in the asset return. The idea of frame of reference or perspective is important here. We are always in the present, which means that the future probability distributions are viewed from the present, and not from some point in the future. What happens in time period  $t_n$  is viewed from the start period (the present)  $t_0$ . This means that the expected returns and variances for each time period are cumulative, where  $t_n < t_{n+1}$  with each period starting at  $t_0$ . A non-cumulative view would be with each period  $t_{n+1}$  starting at  $t_n$ . This cumulative view embeds the effect of compound interest into the statistics. This means that a causal link between the time periods is directly included into the optimization. While it makes the solution easier to determine, it increases the initial complexity of formulating the problem.

To express the expected returns of the assets in terms of the individual asset class weights  $(\omega_{01}, \omega_{02}, \dots, \omega_{0n})$ , we use a superscript/subscript notation such that  $\omega_{ts}$  is the weight of asset  $s$  at time  $t$  (see Appendix 1, Variable Definitions). We then find the relationship between  $E(A_1^0), E(A_2^1), E(A_3^2), \dots, E(A_t^{t-1})$  and  $(\omega_{01}, \omega_{02}, \dots, \omega_{0n})$ .

It is easy to see that for time period 1:

$$E(A_1^0) = \omega_{01}E(\alpha_{11}^0) + \omega_{02}E(\alpha_{12}^0) + \dots + \omega_{0n}E(\alpha_{1n}^0) \quad [1.2]$$

where,  $\alpha_{ts}^k$  the return of asset  $s$  at time  $t$  with start time  $k$ .

This can be expressed as:

$$1 + E(A_1^0) = \omega_{01}(1 + E(\alpha_{11}^0)) + \omega_{02}(1 + E(\alpha_{12}^0)) + \dots + \omega_{0n}(1 + E(\alpha_{1n}^0)) \quad [1.3]$$

Based on our definition of  $E(A_t^0)$  it can be easily shown that:

$$1 + E(A_2^0) = [1 + E(A_1^0)][1 + E(A_2^1)]$$

in a generalized form this becomes:

$$1 + E(A_t^0) = \prod_{i=1}^t [1 + E(A_i^{i-1})] \quad [1.4]$$

## Liabilities

The liabilities  $E(L_1), E(L_2), \dots, E(L_t)$  are cash outflows (or inflows if they are negative) for each time period  $t$ . They are the total future cash outflow (inflow) at that point in time (based on the future value of money, not the present value).

While we could treat these cash flows as independent, this would be trivial, and we would lose information in the relationship between the assets and the liabilities. In order to construct a link between the assets and liabilities, we expand the liability cash flow into a number of factors.

Mathematically, we describe the liability as a summation of factors and the liability's exposure to these factors. In time period one, the factors would be of the form:

$$E(L_1) = E(\xi_{11})(1 + E(\varphi_{11})) + E(\xi_{12})(1 + E(\varphi_{12})) + \dots + E(\xi_{1m})(1 + E(\varphi_{1m})) + E(\varepsilon_1) \quad [1.5]$$

where,  $\varphi_{ts}$  is the liability factor return for factor  $s$  at time  $t$ ,  $\xi_{ts}$  is the liability factor weight for factor  $s$  at time  $t$  and  $\varepsilon_t$  is the residual liability weight at time  $t$ .

Selecting factors is subject to a number of conditions. One of these conditions is that they be a priori factors. That is, even though the factor returns are uncertain, the factor exposures must be known a priori – at the beginning of the period. Another condition is that of relevance. The factor must have a non-trivial correlation with at least one of the asset returns. Also, we must be able to forecast the factor's rate of return. There is no point in selecting a factor where we have no ability to forecast its behavior. Finally, we attach an independence condition in order to add a degree of rigor to the selection of factors. If independent (or near independent) factors cannot be selected, then the exposure should be left in the residual term.

Examples of  $\varphi_{ts}$  would be:

- $\varphi_{t1}$     inflation rate over period  $t$
- $\varphi_{t2}$     real (after inflation) GDP growth over period  $t$ .
- $\varphi_{t3}$     sector GDP rate normalized to real GDP.
- $\varepsilon_t$     residual term.

Most cases will likely involve inflation and the residual.

In time period two, our expected liability would have the form:

$$E(L_2) = E(\xi_{21})(1 + E(\varphi_{11}))(1 + E(\varphi_{21})) + E(\xi_{22})(1 + E(\varphi_{12}))(1 + E(\varphi_{22})) + \dots + E(\xi_{2m})(1 + E(\varphi_{1m}))(1 + E(\varphi_{2m})) + E(\varepsilon_2)$$

and therefore, in time period  $t$ , it would be of the form:

$$E(L_t) = \sum_{l=1}^m [E(\xi_{tl}) \prod_{k=1}^t (1 + E(\varphi_{kl}))] + E(\varepsilon_t) \quad [1.6]$$

The factor weights  $\xi_{ts}$ , are assumed to be independent of the factor returns.

### Combining Assets and Liabilities

Since we want to express  $E(T_t)$  in terms of  $T_0$ , by using equation [1.4] we can set equation [1.1] to the following form:

$$E(T_t) = \left[ [1 + E(A_t^0)] - \sum_{j=1}^t E(L_j)(1 + E(A_t^j)) \right] T_0 \quad [1.7]$$

This is equivalent to creating a negative portfolio every time a cash flow occurs.

This is analogous to a couple of concepts used in quantum mechanics. First, we are creating a virtual negative portfolio. In particle physics a negative virtual particle is created to represent an attracting force. Our method is similar in concept, in that the negative portfolios do not actually exist, but are rather mathematical constructs. The second concept, is the idea of observance defining reality. In quantum mechanics, we do not know what a particle does while travelling between two points, rather we only know its point and time of arrival. The point and time of arrival is defined by when we observe the event to occur. With our method, we use the same concept. We do not care what form the portfolio takes at any and every point in time, rather we are only concerned with its profile when a constraint occurs. The constraints are limited observations at given points in time (ie: does the probability exceed a certain level for a certain balance at a point in time). We are then left with a very tightly constrained amount of information with all extraneous data eliminated.

This method has enormous advantages as well as a disadvantage. The main advantages are the calculations increase linearly over time, and it permits non-normality in the return expectations while still permitting correlations. The first advantage enables us to optimize over any reasonable time frame or group of asset classes with precise statistics at each point in time. Therefore, the statistics in period one can be just as precise as they are in period 30, with only a linear increase in computing time. The second advantage allows us to include important correlations between asset classes and other factors such as inflation while not requiring the asset to behave normally. Our only constraint is that a pair of underlying random variables must have reasonably well behaved probability densities if we want to determine their correlations. The main disadvantage is that we lose path-dependency. Therefore, we cannot include instruments such as mortgage-backed securities and look-back options.

The asset and liability factor returns are assumed to follow a Markov Process. In our case, the asset returns at  $t$  are independent of their returns at  $t-1$ , and are only dependent upon the

portfolio value at t-1. The liability factor returns at t are also assumed to be independent of the factor returns at t-1 as well as the asset returns at t-1. These independence (or orthogonality) relationships of the random variables are summarised as:

$$\left\{ \begin{matrix} \alpha_{11}, \alpha_{12}, \dots, \alpha_{1n} \\ \varphi_{11}, \varphi_{12}, \dots, \varphi_{1m} \end{matrix} \right\} \perp \dots \perp \left\{ \begin{matrix} \alpha_{t1}, \alpha_{t2}, \dots, \alpha_{tm} \\ \varphi_{t1}, \varphi_{t2}, \dots, \varphi_{tm} \end{matrix} \right\} \perp \{\xi_{11}\} \perp \dots \perp \{\xi_{tm}\} \perp \{\varepsilon_1\} \perp \dots \perp \{\varepsilon_t\}$$

When building the model, the user must forecast the asset class returns, liability factor returns, and the correlations between them. The weights must be calculated based on a linear regression of the forecasted liabilities and the liability factor returns. One difficulty arises in selecting the liability factors. Since they provide a link between the underlying behavior of the liability and the assets, a linear relationship must exist not only between the factor and the liability, but with at least one asset as well. In order to keep the underlying statistics stable, the factors themselves must be stable, independent of each other, measurable, meaningful and have a linear relationship with the liability.

## Rebalancing and Reinvesting

While we have a generalized equation in [1.7] we must still determine how to treat the expected asset returns when the start value is other than t=0. These terms

$E(A_2^1), E(A_3^2), \dots, E(A_t^{t-1})$  are dependent upon the rebalancing and reinvestment strategies we adopt. At present, we will use two rebalancing strategies and two reinvesting strategies. Our two rebalancing strategies are Buy-and-Hold and Constant Mix, while our two reinvesting are Perpetual and Maturing.

A Buy-and-Hold strategy is characterized by an initial mix (eg: 60/20/20 stocks/bonds/bills) that is bought and then held. No matter what happens to the relative values, no rebalancing is done.

A Constant Mix strategy maintains the relative weights of the various asset classes as a constant proportion of wealth. An initial mix is set, and whenever the relative values of the assets change, purchases and sales are performed to return the portfolio to the initial mix. In general, when rebalancing a constant mix strategy, assets that fall in relative value are purchased and those that rise in relative value are sold. In this case it is necessary to define how frequently the portfolio will be re-balanced (quarterly, annually etc.). Under the model, each rebalancing point then becomes a point at which we check to see how the assets have performed.

In the Perpetual reinvestment strategy we assume all dividends are reinvested into the same asset. More importantly, we also assume that a bond portfolio acts like an index. Since bonds are maturing assets the Perpetual case cannot treat bonds as individual securities, but rather as a basket of assets that are rolled into new bonds.

The Maturing reinvestment strategy still treats equities as perpetual assets but now treats bonds as maturing assets. A reinvestment criterion must be predefined, however. Examples of this criterion include rolling the matured bond into a short-term money market instrument, purchasing a bond that matures at the end of the final period, purchasing the current portfolio or another rational criterion.

Therefore, the four strategies we shall use are:

- e) Buy and Hold Perpetual
- f) Buy and Hold Maturing
- g) Constant Mix Perpetual
- h) Constant Mix Maturing

a) *Buy and Hold Perpetual:*

First we consider the buy and hold rebalancing strategy. Once the initial portfolio is constructed and the asset mix set, no further action is taken. The asset classes are permitted to increase or decrease in relative value. The result is that we need only concern ourselves with setting the initial weights. Also, since no asset class matures, all of our initial assets exist at each point in time. We find that:

$$1 + E(A_t^0) = \omega_{01}[1 + E(\alpha_{t1}^0)] + \omega_{02}[1 + E(\alpha_{t2}^0)] + \dots + \omega_{0n}[1 + E(\alpha_{tn}^0)] \quad [1.8]$$

We determine the values for  $E(A_j^i)$  (where  $i > 0$ ) by utilizing the following relationship:

$$[1 + E(A_t^j)] = \left( \frac{1 + E(A_t^0)}{1 + E(A_j^0)} \right)$$

and therefore find:

$$E(T_t) = \left[ [1 + E(A_t^0)] - \sum_{i=1}^t E(L_j) \left( \frac{1 + E(A_t^0)}{1 + E(A_j^0)} \right) \right] T_0 \quad [1.9]$$

b) *Buy and Hold Maturing*

Once again, we consider the buy and hold rebalancing strategy, but this time there is the possibility of some assets maturing during the investment period. These maturing assets are assumed to be bonds.

As in the perpetual asset case, we must first determine the values of  $1 + E(A_t^0)$  in terms of the asset weights. Unfortunately, it is substantially more complex than in [1.8]. Since an asset matures, it will cease to exist and therefore become cash. The dilemma becomes how is this cash reinvested.

There are a number of possible reinvestment strategies, such as purchasing T-Bills, bonds that mature at the end of the final period or the current portfolio. We will select a method whereby we reinvest the cash so that it exactly replicates the current portfolio. This will result in no changes to the relative weights of the assets. This is equivalent to creating a negative liability. There are several advantages to reinvesting this way, both conceptually and mathematically. Conceptually, it makes sense since the investor is maintaining their portfolio from their initial forecast. Mathematically it makes sense since the positive portfolio created would exactly cancel a negative portfolio. An example is if we had a cash payment in year three and we held some equities and a three-year strip bond. The strip bond matures creating a positive portfolio, while the liability has created a negative portfolio – the two exactly cancel one another.

One problem we have with this method, is that when the asset (bond) matures, new assets (bonds) will be created that do not exist at present. For example, if a bond matures in three years and we invest the proceeds into a one-year bond, then from today's perspective we are purchasing a three-year bond and a three-year forward rate on a one-year bond.

We now have the expected return from our data for each individual bond, and we must combine them with the expected return of the other asset classes. Our problem is if we do this directly, the matrix would be huge. So, we must do it using a factor model where we describe all the returns in terms of a few factors. This is a reasonable approach since the bond yield curve is highly correlated.

Each bond curve becomes a separate asset class. Consider the case where we have  $n$  assets plus one bond asset class, where the bonds in that class have maturities ranging from one to thirty years. So equation [1.2] with one bond curve asset class is of the form:

$$1 + E(A_t^0) = \sum_{s=1}^n \omega_{0s} (1 + E(\alpha_{ts}^0)) + \omega_{0n+1} \sum_{m=1}^M x_m (1 + E(R_{tm}^0)) \quad [1.10]$$

where  $M$  is the maximum number of periods in the yield curve,  $R_{tm}^k$  is the rate of return of the bond maturing at  $m$  as of time  $t$  with an initial start date  $k$ , and  $x_m$  is the percentage weight of bonds maturing at  $m$  as of the initial start date, with the constraint:

$$\sum_{m=1}^M x_m = 1$$

Expanding [1.10] into a matrix form, we can see what is known and unknown for each time period (where we assume  $M$  to be 30).

$$\left( \begin{array}{cccc|cccc} \alpha_{1,1}^0 & \alpha_{1,2}^0 & \cdots & \alpha_{1,n}^0 & R_{1,1}^0 & R_{1,2}^0 & R_{1,3}^0 & \cdots & R_{1,30}^0 \\ \alpha_{2,1}^0 & \alpha_{2,2}^0 & \cdots & \alpha_{2,n}^0 & ? & R_{2,2}^0 & R_{2,3}^0 & \cdots & R_{2,30}^0 \\ \alpha_{3,1}^0 & \alpha_{3,2}^0 & \cdots & \alpha_{3,n}^0 & ? & ? & R_{3,3}^0 & \cdots & R_{3,30}^0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \alpha_{30,1}^0 & \alpha_{30,2}^0 & \cdots & \alpha_{30,n}^0 & ? & ? & ? & \cdots & R_{30,30}^0 \end{array} \right)$$

There are three separate areas of the bond matrix; the diagonal, above the diagonal and below the diagonal:

- 1) The diagonal is already determined, since it is today's yield curve.
- 2) The area above the diagonal contains the forecasted yield curves.
- 3) The area below the diagonal contains the returns of cash obtained from matured bonds.

Finding values for the area above the diagonal requires generating a series of yield curve forecasts. Since the present yield curve already provides us with a source of expected values for these rates we will use them. One can of course insert one's own expectations.

For our purposes, we find the expected values by the following:

The price of a discount bond maturing at  $m$  as of time  $t$  is:

$$P_{tm} = \frac{1}{(1 + Y_{tm})^{m-t}} \quad [1.11]$$

where  $Y_{tm}$  is the yield of bond  $m$  at time  $t$ .

Therefore, the rate of return of a bond would be:

$$1 + E(R_{tm}^k) = \frac{E(P_{tm})}{E(P_{km})} = \frac{(1 + E(Y_{km}))^{m-k}}{(1 + E(Y_{tm}))^{m-t}} \quad [1.12]$$

where:  $m \geq t \geq k$

Note that  $\rightarrow E(Y_{0m}) = Y_{0m}$  no uncertainty

Also that  $E(R_{mm}^0) = R_{mm}^0$  (no uncertainty)

So,  $P_{23}$  is the price of a 3-year bond 2 years from now, and  $E(R_{23}^0)$  is the expected return of that bond from today to 2 years hence.

The area below the diagonal is more complex. The bond has ceased to exist, but we still have an expected return. Instead of trying to remove this term from the calculation we simply reset it such that the expected return of the expired bond is equivalent to the expected return of the

whole portfolio. This greatly simplifies the calculations, and is consistent with our previous discussion on reinvesting.

We cannot set each row in our matrix equivalent, however. The unknowns in each row are not equivalent since they are the cumulative expected returns of the corresponding bonds. What are equivalent are the 1-period expected returns. Therefore, we have:

$$1 + E(R_{t,1}^{t-1}) = 1 + E(R_{t,2}^{t-1}) = \dots = 1 + E(R_{t,t-1}^{t-1})$$

Then we can calculate the unknowns in the matrix iteratively by:

$$1 + E(R_{t,m}^0) = (1 + E(R_{t-1,m}^0))(1 + E(R_{t,m}^{t-1})) \quad [1.13]$$

Note that only the initial bond weights are explicitly calculated. There is a difference between the total relative weights, and our explicit weights. When an asset's return exceeds another's, the weights change automatically. We capture that change in our cumulative expected returns, not in the explicit weights. We don't know what the weights of the assets are after the initial time period. They can be calculated, but they are not calculated explicitly, since all the information is captured in the cumulative returns.

To calculate the one-period expected return (from time t-1 to time t), remember that the portfolio depends only on the assets that haven't expired. Namely, all the equities and the bonds maturing from time t to M. Therefore, the one-period expected return of the portfolio equals the cumulative expected return from time 0 to t divided by the cumulative expected return from time 0 to t-1:

$$1 + E(R_{t,m}^{t-1}) = \frac{\left[ \sum_{i=1}^n \omega_i (1 + E(\alpha_{t,i}^0)) + \sum_{j=t}^M \omega_{n+1} x_j (1 + E(R_{t,j}^0)) \right]}{\left[ \sum_{i=1}^n \omega_i (1 + E(\alpha_{t-1,i}^0)) + \sum_{j=t}^M \omega_{n+1} x_j (1 + E(R_{t-1,j}^0)) \right]} \quad [1.14]$$

where  $m=1, \dots, t-1$ .

Therefore, inserting into [1.13] we have:

$$1 + E(R_{t,m}^0) = (1 + E(R_{t-1,m}^0)) \cdot \frac{\left[ \sum_{i=1}^n \omega_i (1 + E(\alpha_{t,i}^0)) + \sum_{j=t}^M \omega_{n+1} x_j (1 + E(R_{t,j}^0)) \right]}{\left[ \sum_{i=1}^n \omega_i (1 + E(\alpha_{t-1,i}^0)) + \sum_{j=t}^M \omega_{n+1} x_j (1 + E(R_{t-1,j}^0)) \right]} \quad [1.15]$$

All the values of  $R_{m}^0$  can be found by calculating [1.15] iteratively, from  $t= 1$  to M.

Finally, we expand the number of bond curves to  $b$  bond asset classes. Note that since each bond curve is a separate asset class, they may have different curve lengths. Therefore, let  $J^i$  be the set of indices representing the maturity year of the bonds we considered from the  $i$  bond asset class. So generalizing [1.14] into  $b$  bond asset classes we get:

$$1 + E(A_t^0) = \sum_{s=1}^n \omega_s (1 + E(\alpha_{ts}^0)) + \omega_{n+1} \sum_{m=1, m \in J^1}^M x_m^{(1)} (1 + E(R_{tm}^{(1)0})) + \cdots + \omega_{n+b} \sum_{m=1, m \in J^b}^M x_m^{(b)} (1 + E(R_{tm}^{(b)0})) \quad [1.16]$$

where,

$$1 + E(R_{t,m}^{(b)0}) = (1 + E(R_{1,m}^{(b)0})) \cdots (1 + E(R_{m,m}^{(b)0})) \cdots (1 + E(R_{m+1,m}^{(b)0})) \cdots (1 + E(R_{t,m}^{(b)0})) \quad [1.17]$$

$$1 + E(R_{k,m}^{(b)k-1}) = \frac{\left[ \sum_{i=1}^n \omega_i (1 + E(\alpha_{k,i}^0)) + \sum_{j=k, j \in J^1}^M \omega_{n+1} x_j (1 + E(R_{k,j}^{(1)0})) + \cdots + \sum_{j=k, j \in J^b}^M \omega_{n+b} x_j (1 + E(R_{k,j}^{(b)0})) \right]}{\left[ \sum_{i=1}^n \omega_i (1 + E(\alpha_{k-1,i}^0)) + \sum_{j=k, j \in J^1}^M \omega_{n+1} x_j (1 + E(R_{k-1,j}^{(1)0})) + \cdots + \sum_{j=k, j \in J^b}^M \omega_{n+b} x_j (1 + E(R_{k-1,j}^{(b)0})) \right]} \quad [1.18]$$

for  $k$  from  $m+1$  to  $t$ .

These are then substituted into equation [1.9] from the buy and hold perpetual case:

$$E(T_t) = \left\{ (1 + E(A_t^0)) - \sum_{j=1}^t E(L_j) \left( \frac{1 + E(A_t^0)}{1 + E(A_j^0)} \right) \right\} T_0 \quad [1.9]$$

c) *Constant Mix Perpetual*

Suppose there is no expiry date for each asset. The constant mix strategy is one where we rebalance the weights of the assets so that the weights remain constant over time. This constant weight can be expressed as:

$$E(A_t^{i-1}) = \omega_{01} E(\alpha_{t1}) + \omega_{02} E(\alpha_{t2}) + \cdots + \omega_{0n} E(\alpha_{tn}) \quad \text{for } i = 1, \dots, t$$

Inserting this format into equation [1.1] we have:

$$E(T_t) = \left[ [1 + E(A_t^0)] - \sum_{j=1}^t E(L_j) (1 + E(A_j^0)) \right] T_0$$

where,

$$1 + E(A_t^0) = \prod_{i=1}^t [1 + E(A_i^{i-1})]$$

$$= \prod_{i=1}^t (1 + \omega_{01}E(\alpha_{i1}) + \omega_{02}E(\alpha_{i2}) + \dots + \omega_n E(\alpha_{in}))$$

and,

$$1 + E(A_t^j) = \prod_{i=j+1}^t [1 + E(A_i^{i-1})]$$

$$= \prod_{i=j+1}^t (1 + \omega_{01}E(\alpha_{i1}) + \omega_{02}E(\alpha_{i2}) + \dots + \omega_n E(\alpha_{in}))$$

**[1.20]**

c) *Constant Mix Maturing*

Suppose we include bonds that mature before the end of the investment horizon. We still want to keep the constant mix strategy, but a problem exists when one of the assets expires resulting in the total weights of the remaining assets becomes less than one.

Consider the following special case. Suppose only asset q expires at time  $t=Q$ . Then,

$$E(A_i^{i-1}) = \omega_{0,1}E(\alpha_{i,1}) + \dots + \omega_{0,q-1}E(\alpha_{i,q-1}) + \omega_{0,q}E(\alpha_{i,q}) + \omega_{0,q+1}E(\alpha_{i,q+1}) + \dots + \omega_{0,n}E(\alpha_{i,n})$$

for  $i \leq Q$ .

When a security matures at time t it ceases to exist, however. Since,  $E(\alpha_{Q+1,q})$ ,  $E(\alpha_{Q+2,q})$ , ...,  $E(\alpha_{t,q})$  are undefined, we do the following:

$$E(A_i^{i-1}) = \frac{\omega_{0,1}}{1 - \omega_{0,q}} E(\alpha_{i,1}) + \dots + \frac{\omega_{0,q-1}}{1 - \omega_{0,q}} E(\alpha_{i,q-1}) + \frac{\omega_{0,q+1}}{1 - \omega_{0,q}} E(\alpha_{i,q+1}) + \dots + \frac{\omega_{0,n}}{1 - \omega_{0,q}} E(\alpha_{i,n})$$

for  $Q+1 \leq i \leq t$ .

In this way, the sum of the weights is one and the mix is 'relatively' constant. However, to avoid dividing by zero we have to assume there is at least one asset that never matures in the period and has positive weight.

So far we've structured or solution for only one maturing bond, now we will generalize the solution.

In general, define:

$I_t$  = set of indices representing the assets which **have not** expired before time  $t$ .

$J_t$  = set of indices representing the assets which **have** expired before time  $t$ .

Therefore,  $I_t \cup J_t = \{1, \dots, n\}$  for all  $t$ .

By definition, at  $t=1$ ,  $J_1=0$  and  $I_1=\{1, \dots, n\}$

So for the one-asset-expiring example above,

$$I_0=I_1=\dots=I_Q = \{1, 2, \dots, n\}$$

$$J_0=J_1=\dots=J_Q = \emptyset$$

$$I_{Q+1}=I_{Q+2}=\dots=I_H = \{1, 2, \dots, q-1, q+1, \dots, n\}$$

$$J_{Q+1}=J_{Q+2}=\dots=J_H = \{n\}$$

Where,  $n$  is the number of assets, and  $H$  is the end of the Horizon period.

Assume for all time periods  $i$ , there is an  $l \in I_t$  such that  $\omega_{i,l} > 0$ .

This is to guarantee  $\sum_{l=1, l \in J_i} \omega_{0l} < 1$  for all  $i$ . Then:

$$E(A_t^{t-1}) = \sum_{i=1, k \in I_t}^n \frac{\omega_{0k} E(\alpha_{ik})}{1 - \sum_{l=1, l \in J_t}^n \omega_{0l}} \quad [1.21]$$

Hence, combining [1.21] and [1.1] we obtain:

$$E(T_t) = \left[ [1 + E(A_t^0)] - \sum_{j=1}^t E(L_j) (1 + E(A_t^j)) \right] T_0$$

where,

$$E(A_t^0) = \prod_{i=1}^t \left( 1 + \sum_{i=1, k \in I_t}^n \frac{\omega_{0k} E(\alpha_{ik})}{1 - \sum_{l=1, l \in J_t}^n \omega_{0l}} \right)$$

and

$$E(A_t^j) = \prod_{i=j+1}^t \left( 1 + \sum_{i=1, k \in I_t}^n \frac{\omega_{0k} E(\alpha_{ik})}{1 - \sum_{l=1, l \in J_t}^n \omega_{0l}} \right)$$

and therefore obtain:

$$E(T_t) = \left\{ \prod_{i=1}^t \left( 1 + \sum_{k=1, k \in I_i}^n \frac{\omega_{0k} E(\alpha_{ik})}{1 - \sum_{l \in J_i}^n \omega_{0l}} \right) - \sum_{j=1}^t E(L_j) \prod_{i=j+1}^t \left( 1 + \sum_{k=1, k \in I_i}^n \frac{\omega_{0k} E(\alpha_{ik})}{1 - \sum_{l \in J_i}^n \omega_{0l}} \right) \right\} T_0 \quad [1.22]$$

As in the Buy and Hold maturing model, the expected return for existing bonds can be estimated from our interest rate model. However, in this case we don't need to get the expected return for matured bonds.